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B.M.S COLLEGE FOR WOMEN, AUTONOMOUS
BENGALURU -560004
SEMESTER END EXAMINATION – APRIL/ MAY 2023

M.Sc. Mathematics – III Semester

FUNCTIONAL ANALYSIS

Course Code: MM302T **QP Code: 13002**
Duration: 3 Hours **Max. Marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. a) Let $T: N \to N'$ be a linear transformation of a normed linear space N into another normed linear space N' then prove the following

   i) T is continuous on N.
   ii) T is continuous at origin .
   iii) T is bounded if $k > 0$ such that $\|Tx\| \leq k\|x\|$ $\forall x \in N$.

   b) Define Banach space. Prove that $\mathbb{C}^n$ is Banach space under the norm $\|x\| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$. **(8+6)**

2. a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in a quotient space N/M is defined by $\|x + M\| = \inf\{\|x + M\|: m \in M\}$ then prove that N/M is normed linear space and a Banach space.
   b) Prove Minkowski's inequality in l_p^n space where $1 \leq p < \infty$. **(8+6)**

3. a) State and prove Hahn-Banach theorem for real normed linear spaces.
   b) Show that there is a natural embedding of N into N^{**} obtained by isometric isomorphism $\phi: N \to N^{**}$ defined by $\phi(x) = F_x$where $F_x(f) = f(x)$, $\forall f \in N^*$, $x \in N$. **(8+6)**

4. a) State and prove Banach Steinhus theorem.
   b) If E is projection on a Banach space B and if M and N are range and null space then prove that M and N are closed linear subspace of B such that $B = M \oplus N$. **(8+6)**

5. a) Define Hilbert space. State and prove Schwartz inequality and Bessel's inequality.
   b) Prove that translation on a Hilbert space preserves convexity and closure. **(8+6)**

6. a) Let $\{e_1, e_2, \ldots\ \}$ be a finite orthonormal set in H if $x \in H$ then prove that

i) $\sum_{i=1}^{n}[\langle x, e_i\rangle]^2 \leq \|x\|^2$

ii) $x - \sum_{i=1}^{n}[\langle x, e_i\rangle]e_i \ \perp\ e_j,\ \forall j$

b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

c) Prove that the inner product is a continuous function on $X \times X$. **(6+4+4)**

7. a) Prove that the adjoint operator satisfies the following

i. $(T_1 + T_2)^* = T_1^* + T_2^*$.

ii. $\|T^*T\| = \|TT^*\|$.

iii. If T is non-singular $\Longrightarrow T^*$ is also.

iv. $(T_1T_2)^* = T_2^*T_1^*$.

b) An idempotent operator on a Hilbert space H is projection on H if and only if it is normal. **(10+4)**

8. a) Let H be a Hilbert space and T be a positive operator on H then prove that $I + T$ is non-singular.

b) Prove that for an antilinear norm preserves isometric isomorphism between H and $H^\star$.

(8+8)

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